Linear Algebra Problem Set 1 Solutions

1. Through Gaussian Elimination the matrix reduces to:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ -1 & -2 & -2 & -5 \\ 3 & 5 & 4 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, $x_1 = -1, x_2 = 2, x_3 = 1$

Through Gauss-Jordan Elimination the matrix reduces to:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 1 & -5 & 0 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & -8 & -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 & 3 \\ 0 & 8 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 + x_3 + x_4 = 3$

$8x_2 + x_3 = -2$

There will be two free variables:

$x_4 = t, x_3 = s,\ x_2 = -\frac{1}{4} - \frac{s}{8},\ x_1 = \frac{15}{4} - \frac{5}{8}s - t$

2. After Gauss Elimination the matrix becomes:

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{bmatrix}$$

For the system to be consistent $c - a - b = 0 \Rightarrow c = a + b$

3.

a) $CA = \begin{bmatrix} 11 & 3 & 1 \\ 6 & 1 & 8 \end{bmatrix}$

b) AB is not possible

c) $B^T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$

d) $B^T CA + I = \begin{bmatrix} 28 & 7 & 10 \\ 11 & 3 & 1 \\ 27 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 29 & 7 & 10 \\ 11 & 4 & 1 \\ 27 & 8 & -4 \end{bmatrix}$

4. $c + p = 130$
$2c + 4p = 464$

You can solve using Gaussian elimination, substitution or elimination method. Either way you should get that there were 102 pigs and 28 chickens.

5. The trace of a matrix is the sum of the diagonal entries, so

$$tr(A+B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + ... + (a_{nn} + b_{nn})$$
$$= (a_{11} + a_{22} + ... + a_{nn}) + (b_{11} + b_{22} + ... + b_{nn})$$
$$tr(A) + tr(B)$$

6. .

a) Always true: $A = nxm \Rightarrow A^T = nxm \Rightarrow AA^T = mxm, A^T A = nxn$, both are square matrices so the trace is defined for both.

b) Always true:

$$(AA^T)_{ii} = a_{i1}a_{1i}{}^T + a_{i2}a_{2i}{}^T + ... + a_{in}a_{ni}{}^T$$
$$= a_{i1}a_{i1} + a_{i2}a_{i2} + ... + a_{in}a_{in}$$
$$= a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2$$

The trace is the sum of the main diagonal elements so,

$$tr(AA^T) = \sum_{i=1}^{m} a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2$$

This is the sum of the squares of all the terms in matrix A

$$(A^T A)_{ii} = a_{i1}{}^T a_{1i} + a_{i2}{}^T a_{2i} + ... + a_{im}{}^T a_{ni}$$
$$= a_{1i}a_{1i} + a_{2i}a_{2i} + ... + a_{ni}a_{ni}$$
$$= a_{1i}{}^2 + a_{2i}{}^2 + ... + a_{ni}{}^2$$
$$tr(A^T A) = \sum_{i=1}^{n} a_{1i}{}^2 + a_{2i}{}^2 + ... + a_{mi}{}^2$$

This is also the sum of the squares of all the terms in matrix A

Hence $tr(AA^T) = tr(A^T A)$

c) False: Need to just show one counterexample

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

d) Always true: Assume first row of a matrix A has all zero entries then the elements in the first row of the product AB would be:

$$[AB]_{1i} = a_{11}b_{1i} + a_{12}b_{2i} + ... + a_{1n}b_{ni} = 0 * b_{1i} + 0 * b_{2i} + ... + 0 * b_{ni} = 0$$