Practice Problems Exam 2 Solutions

1. Consider the bases $B = [u_1, u_2]$ and $B' = [v_1, v_2]$ for R^2 where
   $u_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $v_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$ $v_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$
   a) Find the transition matrix B' to B.
   b) Find the transition matrix from B to B'.
   c) Compute the coordinate vector $[w]_B$ where $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. Let $u = (1,2,3)$, $v = (-3,4,-1)$ and $w = (0,-1,1)$. Find
   a) $u - 2w$
   b) $u \cdot v$
   c) $w \times v$
   d) $\|3u + w\|$
   e) $proj_u w$
   f) A unit vector in the opposite direction of $u + v$
   g) A vector orthogonal to both u and v.
   h) Find scalars k, such that $\|kw\| = 12$.
   i) Find the area of the parallelogram determined by u and v.
   j) Find the volume of the parallelepiped determined by u, v and w.

3. If $u \cdot v = u \cdot w$, is $v = w$? (Assume they are all non-zero vectors).

4. Let $P_1(2,0,1)$, $P_2(-1,5,3)$, $P_3(5,2,0)$. Find the area of the triangle having vertices P_1, P_2, P_3.
5. Find $u \cdot v$ given that $\|u + v\| = 1$ and $\|u - v\| = 5$.
6. Are the following linearly independent?
   a) {(1,3,2), (-4,2,1), (5,-1,0)}
   b) $\{3 - x + x^2, x^2 + 1, 7x - 2\}$
   c) {(2,4), (0,9),(-1,4)}

7. Compute $\|f\|$ given the inner product is defined as $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ with $f(x) = e^x$.

8. Determine whether or not the following sets are vector spaces under the given operations.
   a) The set of all triples of real numbers (x, y, z) with the operations
   $(x, y, z) + (x', y', z') = (y + y', x + x', z + z')$ and $k(x, y, z) = (kx, ky, kz)$

b) The set of all 2x2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with addition defined by

$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and scalar multiplication defined by

$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$.

9. Determine which if the following are subspaces.
   a) All vectors in R^3 of the form (a,b,c) where a, b and c are integers.
   b) All polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ for which $a_0 = 0$.
   c) All invertible 2x2 matrices.

10. Determine whether the solution space of the system $Ax = 0$ is a line through the origin, a plane through the origin or the origin only. Give the equation for the line or plane.

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 1 & -5 \\ 1 & 4 & 3 \end{bmatrix}$$

11. Are the following bases?
    a) {(3,0,2), (-1,5,3), (2,-1,5), (1,1,1)} in R^3?
    b) $\{x^2, x, 7+x\}$ in P_2?
    c) {(1,-1), (1,3)} in R^2?

12. Determine the dimension and basis for the solution space of the following system.

$$x_1 - 2x_2 - x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

13. Given the following matrix, find bases for its row space, column space and nullspace. Determine the rank and nullity. $A = \begin{bmatrix} 4 & 3 & -1 \\ -1 & 2 & 3 \\ 5 & 2 & -3 \end{bmatrix}$

14. Which of the following sets are orthogonal with respect to the Euclidean norm? Are they orthonormal?
    a) {(1,-1,1), (2,0,-2), (-1,2,3)}
    b) $\{(1/\sqrt{2}, -1/\sqrt{2}), (1/\sqrt{2}, 1/\sqrt{2})$

15. Use Gram-Schmidt process to transform the following basis into an orthonormal set. {(2,1,0), (-1,0,4), (3,-2,1)}