Practice Problems Exam 2 Solutions

1. Consider the bases $B=[u_1,u_2]$ and $B'=[v_1,v_2]$ for R^2 where
$u_1=\begin{bmatrix}1\\-3\end{bmatrix}$ $u_2=\begin{bmatrix}-2\\4\end{bmatrix}$ $v_1=\begin{bmatrix}-7\\9\end{bmatrix}$ $v_2=\begin{bmatrix}-5\\7\end{bmatrix}$
   a) Find the transition matrix B' to B.
   $$\begin{bmatrix}1 & -2 & -7 & -5\\ -3 & 4 & 9 & 7\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 0 & 5 & 3\\ 0 & 1 & 6 & 4\end{bmatrix}$$
   $$P=\begin{bmatrix}5 & 3\\ 6 & 4\end{bmatrix}$$
   b) Find the transition matrix from B to B'.
   $$\begin{bmatrix}-7 & -5 & 1 & -2\\ 9 & 7 & -3 & 4\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 0 & 2 & -3/2\\ 0 & 1 & 3 & 5/2\end{bmatrix}$$
   $$P^{-1}=\begin{bmatrix}2 & -3/2\\ 3 & 5/2\end{bmatrix}$$
   c) Compute the coordinate vector $[w]_B$ where $w=\begin{bmatrix}2\\1\end{bmatrix}$
   $$=\begin{bmatrix}5 & 3\\ 6 & 4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}13\\16\end{bmatrix}$$

2. Let $u=(1,2,3)$, $v=(-3,4,-1)$ and $w=(0,-1,1)$. Find
   a) $u-2w=(1,4,1)$
   b) $u\cdot v=2$
   c) $w\times v(-3,-3,-3)$
   d) $\|3u+w\|=\sqrt{134}$
   e) $proj_u w=\left(\frac{1}{14},\frac{1}{7},\frac{3}{14}\right)$
   f) A unit vector in the opposite direction of $u+v=\left(\frac{2}{\sqrt{44}},\frac{-6}{\sqrt{44}},\frac{-2}{\sqrt{44}}\right)$
   g) A vector orthogonal to both u and v $=(-14,-8,10)$.
   h) Find scalars k, such that $\|kw\|=12$. $k=\frac{12}{\sqrt{2}}$
   i) Find the area of the parallelogram determined by u and v $=6\sqrt{10}$.
   j) Find the volume of the parallelepiped determined by u, v and w $=18$.

3. If $u\cdot v=u\cdot w$, is $v=w$? (Assume they are all non-zero vectors). No, let $u=(1,1,1)$, $v=(1,0,0)$ and $w=(0,0,1)$.

4. Let $P_1(2,0,1)$, $P_2(-1,5,3)$, $P_3(5,2,0)$. Find the area of the triangle having vertices P_1, P_2, P_3. $A = \frac{1}{2}\|P_1P_2 \times P_2P_3\| = \frac{1}{2}\sqrt{531}$

5. Find $u \cdot v$ given that $\|u+v\| = 1$ and $\|u-v\| = 5$.

   $\|u+v\|^2 = (u+v)\cdot(u+v) = \|u\|^2 + \|v\|^2 + 2(u \cdot v)$ and

   $\|u-v\|^2 = (u-v)\cdot(u-v) = \|u\|^2 + \|v\|^2 - 2(u \cdot v)$

   Subtracting the two equations gives:

   $\|u+v\|^2 - \|u-v\|^2 = 4(u \cdot v) \Rightarrow$

   $$\frac{\|u+v\|^2 - \|u-v\|^2}{4} = \frac{(1)^2 - (5)^2}{4} = u \cdot v = -6$$

6. Are the following linearly independent?
   a) {(1,3,2), (-4,2,1), (5,-1,0)} $\det\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix} \neq 0$ so these vectors are linear independent.
   b) $\{3 - x + x^2, x^2 + 1, 7x - 2\}$ $W = \det\begin{bmatrix} 3 - x + x^2 & x^2 + 1 & 7x - 2 \\ -1 + 2x & 2x & 7 \\ 2 & 2 & 0 \end{bmatrix} \neq 0$ therefore linearly independent
   c) {(2,4), (0,9),(-1,4), too many vectors so linearly dependent

7. Compute $\|f\|$ given the inner product is defined as $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ with $f(x) = e^x$.

   $$\langle f, f\rangle = \int_0^1 f(x)f(x)dx = \int_0^1 e^{2x}dx = \frac{e^{2x}}{2} = \frac{e^2}{2} - \frac{1}{2}$$

   $$\|f\| = \sqrt{\frac{e^2}{2} - \frac{1}{2}}$$

8. Determine whether or not the following sets are vector spaces under the given operations.
   a) The set of all triples of real numbers (x, y, z) with the operations
      $(x, y, z) + (x', y', z') = (y + y', x + x', z + z')$ and $k(x, y, z) = (kx, ky, kz)$
      No, fails Axioms 3 (fails others, but as soon as one fails it is not a vector space)

b) The set of all 2x2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with addition defined by

$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and scalar multiplication defined by

$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$.

Satisfies all axioms (would need to show that all ten Axioms are satisfied) so yes it is a vector space.

9. Determine which if the following are subspaces.
   a) All vectors in R^3 of the form (a,b,c) where a, b and c are integers.
      If you add two integers you will still get an integer so it is closed under addition. But if you multiple by a non-integer k, you will not get an integer vector thus it is not closed under scalar multiplication so it is not a subspace
   b) All polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ for which $a_0 = 0$.
      $q(x) = b_1 x + b_2 x^2$ and $r(x) = a_1 x + a_2 x^2$,
      $q(x) + r(x) = (a_1 + b_1)x + (a_2 + b_2)x^2$ so closed under addition
      $kq(x) = kb_1 x + kb_2 x^2$ so closed under scalar multiplication
      Thus forms a subspace
   c) All invertible 2x2 matrices. The addition of two invertible matrices is not necessarily invertible. For example let A=I and B=-I. Then A+B=0 which is not invertible.

10. Determine whether the solution space of the system $Ax = 0$ is a line through the origin, a plane through the origin or the origin only. Give the equation for the line or plane.
    $A = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 1 & -5 \\ 1 & 4 & 3 \end{bmatrix}$ $Ax = 0 \Rightarrow \begin{bmatrix} 2 & -1 & 7 & 0 \\ 0 & 1 & -5 & 0 \\ 1 & 4 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 6 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -44 & 0 \end{bmatrix}$ No free variable, so only trivial solution - the origin

11. Are the following bases?
    a) {(3,0,2), (-1,5,3), (2,-1,5), (1,1,1)} in R^3? No there are four vectors so they can't be linearly independent so they do not form a basis.
    b) $\{x^2, x, 7 + x\}$ in P_2? The Wronskian is not zero so these vectors are linearly independent. P_2 is three-dimension and we have vectors so linearly independence implies they form a basis.
    c) {(1,-1), (1,3)} in R^2? These vectors are linearly independent (the determinant is not zero) therefore they form a basis.

12. Determine the dimension and basis for the solution space of the following system.

$$x_1 - 2x_2 - x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution is $x_3 = t, x_2 = -t, x_1 = -t = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = tv_1$

Dim=1 because one free variable. The basis would be $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = v_1$

13. Given the following matrix, find bases for its row space, column space and nullspace. Determine the rank and nullity.

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -1 & 2 & 3 \\ 5 & 2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 & -1 & 0 \\ -1 & 2 & 3 & 0 \\ 5 & 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution is $x_3 = t, x_2 = -t, x_1 = t$

Basis for rowspace is [1 -2 3], [0 1 1], Basis for column space is $\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

Basis for nullspace is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, so rank(A)=2 and nullity(A)=1

14. Which of the following sets are orthogonal with respect to the Euclidean norm? Are they orthonormal?
   a) {(1,-1,1), (2,0,-2), (-1,2,3)}not orthogonal
   b) $\{(1/\sqrt{2}, -1/\sqrt{2}), (1/\sqrt{2}, 1/\sqrt{2})$ orthogonal and orthonormal

15. Use Gram-Schmidt process to transform the following basis into an orthonormal set.

{(2,1,0), (-1,0,4), (3,-2,1)}

$u_1 = (2,1,0)$

$u_2 = v_2 - proj_{u_1} v_2 = (-1/5, 2/5, 4)$

$u_3 = v_3 - proj_{u_1} v_3 - proj_{u_2} v_3 = (116/81, -232/81, 29/81)$

Normalize each by dividing by the norm to form an orthonormal set