MATH 3323 Linear Algebra
Practice Problems Exam 1

1. Solve the system

$$\begin{cases} \quad\; x_2 + 4x_3 = -5 \\ x_1 + 3x_2 + 5x_3 = -2 \\ 3x_1 + 7x_2 + 7x_3 = 4 \end{cases}$$

$x_1 = 13 + 7w, x_2 = -5 - 4w, x_3 = w$

2. Give an example of an inconsistent system with two equations and three unknowns.

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 2x_3 = 3 \end{cases}$$

3. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that $Ax = b$ does not have a solution for all possible b and determine the set of all b for which $Ax = b$ does have a solution.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 - 2b_2 \end{bmatrix}$$

4. Determine if the following system has a non-trivial solution.

$$\begin{cases} x_1 - 3x_2 + 7x_3 = 0 \\ -2x_1 + x_2 - 4x_3 = 0 \\ x_1 + 2x_2 + 9x_3 = 0 \end{cases}$$

Reduces to: $\begin{bmatrix} 1 & -3 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 12 \end{bmatrix}$ so the only solution is $x_1 = x_2 = x_3 = 0$.

5. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Compute $(3A + B)^T$, CA, $(C^{-1})^T$, $(B - A)C$ and $tr(C^3)$ if possible. If not, explain.

$$(3A + B)^T = \begin{bmatrix} 13 & 13 \\ -5 & -19 \\ -2 & 3 \end{bmatrix}$$

$$CA = \begin{bmatrix} 6 & -10 & 3 \\ 0 & -5 & 4 \end{bmatrix}$$

$$(C^{-1})^T = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$(B - A)C$ not possible

$tr(C^3) = -22$

6. If the second column of B is all zeros, what can you say about the second column of AB? Explain your reasoning.
   The second column of AB = A*second column of B=0. This implies the second column of AB is 0.

7. Is the following equality true? $(A+B)(A-B) = A^2 - B^2$
   $(A+B)(A-B) = A^2 - AB + BA - B^2$, since $AB \neq BA$ always then this is not true.

8. Show that if $ABCD$ is invertible, then $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$.
   $(ABCD)D^{-1}C^{-1}B^{-1}A^{-1} = ABCDD^{-1}C^{-1}B^{-1}A^{-1} = ABCIC^{-1}B^{-1}A^{-1} = ABCC^{-1}B^{-1}A^{-1} =$
   $ABIB^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$

   Since $(ABCD)D^{-1}C^{-1}B^{-1}A^{-1} = I$ then $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$.

9. Find the inverse of $A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$ if it exists. If not, explain.

   None, det = 0

10. Compute the following determinant using cofactor expansion.
    $$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 1$$

11. Compute the following determinants using known theorems (not cofactor expansion). Justify your answers
    $$\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (3)(-2)(1)(2) = -12$$

$$\begin{vmatrix} 0 & 2 & 4 & -2 \\ 1 & 3 & -1 & 4 \\ 0 & 0 & 3 & 6 \\ 1 & 3 & -1 & 6 \end{vmatrix} = (-)\begin{vmatrix} 1 & 3 & -1 & 4 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 3 & 6 \\ 1 & 3 & -1 & 6 \end{vmatrix} = (-)\begin{vmatrix} 1 & 3 & -1 & 4 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (-)(1)(2)(3)(2) = -12$$

12. Let $A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

a) Create an elementary matrix E_1 that will swap R_2 and R_4.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b) Create an elementary matrix E_2 that multiplies R_3 by -6.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Create an elementary matrix E_3 that adds $4R_2$ to R_1.

$$E_3 = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) What is the result of multiplying E_1, E_2 and E_3 (in that order) to A?

$$E_3E_2E_1A = \begin{bmatrix} -4 & -1 & 4 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -6 & 0 & 6 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

13. Solve the system using LU-factorization. State your L and your U matrices.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -8 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -20 \end{bmatrix}$$

$$Ux = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -20 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ -20/3 \end{bmatrix}$$

14. Let $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

a) Find the matrix of cofactors $= \begin{bmatrix} -13 & 2 & 10 \\ 20 & -3 & -15 \\ -12 & 2 & 9 \end{bmatrix}$

b) Find the adjoint of A $= \begin{bmatrix} -13 & 20 & -12 \\ 2 & -3 & 2 \\ 10 & -15 & 9 \end{bmatrix}$

c) Using your answers from about (including what you found in problem 10), write down $A^{-1} = \begin{bmatrix} -13 & 20 & -12 \\ 2 & -3 & 2 \\ 10 & -15 & 9 \end{bmatrix}$ (because det(A)=1).

15. Suppose A and B are 3x3 matrices, and that $\det(A) = 2$ and $\det(B) = -3$. Find each of the following:

a) $\det(BA) = -6$

b) $\det(3B) = -81$

c) $\det(A^{-1}) = 1/2$

16. If A is symmetric, prove $A - AA^T$ is symmetric.

A is symmetric, so $A = A^T$

$(A - AA^T)^T = A^T - (AA^T)^T = A^T - A^{T^T}A^T = A^T - AA^T = A - AA^T$. Therefore $A - AA^T$ is symmetric

17. Find the area of the triangle whose vertices are (1,2), (3,-4) and (-5,-1)

$$\pm\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 1 \\ -5 & -1 & 1 \end{vmatrix} = \pm\frac{1}{2}[(-4-10-3)-(20-1+6)] = \pm\frac{1}{2}[-17-25] = 21$$

18. Find the equation of the plane passing through the points (0,3,1), (0,2,0) and (2,0,2)

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} x & y-2 & z & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 1 \end{vmatrix} = (1)\begin{vmatrix} x & y-2 & z \\ 0 & 1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = (2x + 2(y-2) + 0) - (2z - 2x + 0) = 0$$

$$4x + 2y - 2z = 4$$
$$2x + y - z = 2$$