Linear Algebra Problem Set 2 Solutions

1. $\begin{bmatrix} -3 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -5/31 & 4/31 \\ 4/31 & 3/31 \end{bmatrix}$,

$$A^{-1}b = x = \begin{bmatrix} -6/31 \\ 11/31 \end{bmatrix} \Rightarrow x = -6/11, y = 11/31$$

2.

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Check by showing $AA^{-1} = I$. Details are below.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-5R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{4R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{5R_3+R_1} \xrightarrow{-4R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

3. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -8 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -20 \end{bmatrix}$$

$$Ux = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -20 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ -20/3 \end{bmatrix}$$

4. Assume that there is an inverse and then try and find it. If there is an inverse then:

$$\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}\begin{bmatrix} X & Y \\ W & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

If we can find X, Y, W and Z then we have found the inverse. Do the multiplication on the left and set it equal to the right.

$$\begin{bmatrix} 0X + BW & 0Y + BZ \\ CX + DW & CY + DZ \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$0X + BW = I \Rightarrow BW = I \Rightarrow W = B^{-1}$$
$$0Y + BZ = 0 \Rightarrow BZ = 0 \Rightarrow Z = 0$$
$$CX + DW = 0 \Rightarrow CX + DB^{-1} = 0 \Rightarrow CX = -DB^{-1} \Rightarrow X = -C^{-1}DB^{-1}$$
$$CY + DZ = I \Rightarrow CY + D0 = I \Rightarrow CY = I \Rightarrow Y = C^{-1}$$

So the inverse is:

$$\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} -C^{-1}DB^{-1} & C^{-1} \\ B^{-1} & 0 \end{bmatrix}$$

C and B must be invertible.

5.

a) Let A be a square matrix with an ith row of zeros. Let $B = A^{-1}$. Then $AB = I$. The ith row of AB = ith row of AB

$[AB]ij = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = 0b_{1j} + 0b_{2j} + \ldots + 0b_{nj} = 0$ Therefore $AB \neq I$

Thus a matrix with a row of all zeros is not invertible.

b) A similar argument is used to show if a matrix has a column of zeros it is not invertible. Use the fact that $BA = I$ and reach a contradiction.

6.

a) False: The zero matrix is a counterexample. Any non-invertible matrix will not be able to be written as the product elementary matrices.

b) False: Let $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ are both elementary matrices, but $E_1 E_2 = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$ is not an elementary matrix.

c) True: Multiplying a row of matrix A by a constant and adding it to another row is an elementary row operation and can be written as $E_1 A$. If A is invertible then this product is invertible because an elementary matrix is invertible and the product of invertible matrices is invertible.

d) True: $AB = 0 \Rightarrow A^{-1}AB = A^{-1}0 \Rightarrow IB = 0 \Rightarrow B = 0$

7.

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{bmatrix}$$

After Gauss Elimination the matrix becomes:

$$\begin{bmatrix} 1 & 0 & 0 & -3b_1 - 1/2b_2 - 1/2b_3 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & 1 & 4b_1 - 1/2b_2 + 3/2b_3 \end{bmatrix}$$

The system is consistent for all b's so no restrictions.

$x_1 = -3b_1 - 1/2b_2 - 1/2b_3$

$x_2 = -4b_1 - b_3$

$x_3 = 4b_1 - 1/2b_2 + 3/2b_3$