MATH 3323 Linear Algebra
Solutions Problem Set 3

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1. Let $A = \begin{bmatrix} 1 & 8 & 7 \\ 2 & 10 & 6 \\ 3 & 4 & 5 \end{bmatrix}$
   a) Find the $\det(A) = -64$
   b) Find the matrix of cofactors of A. $C = \begin{bmatrix} 26 & 8 & -22 \\ -12 & -16 & 20 \\ -22 & 8 & -6 \end{bmatrix}$
   c) Find the adjoint of A. $adj(A) = \begin{bmatrix} 26 & -12 & -22 \\ 8 & -16 & 8 \\ -22 & 20 & -6 \end{bmatrix}$
   d) Using your answers from parts a) and c), write down A^{-1}. $A^{-1} = \dfrac{1}{-64}\begin{bmatrix} 26 & -12 & -22 \\ 8 & -16 & 8 \\ -22 & 20 & -6 \end{bmatrix}$

2. Suppose A, B, and C are 2x2 matrices, and that $\det(A) = -3$, $\det(B) = 2$, and $\det(C) = 1/2$. Find each of the following:
   a) $\det(AB) = \det(A)\det(B) = -6$
   b) $\det(C^T) = \det(C) = 1/2$
   c) $\det(B^{-1}) = 1/\det(B) = 1/2$
   d) $\det(2A) = 2^2 \det(A) = -12$
   e) $\det(BA^{-2}C) = \det(B)\det(A^{-2})\det(C) = \det(B)\det(A^{-1})\det(A^{-1})\det(C) = 1/9$

3. Find $\det\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 3 & 0 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix} = 1 * \begin{vmatrix} 1 & 3 & 0 \\ 3 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} - 3 * \begin{vmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} = 1(-17) - 3(-24) = 55$.

4. Use known theorems (not cofactor expansion!) to find each of the following determinants. Justify your answers:
   a) $\det\begin{bmatrix} 1 & -2 & 3 \\ 7 & 1 & 4 \\ -2 & 4 & -6 \end{bmatrix} = 0$ Row 3 is multiple of row 1.

b) $\det\begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & 4 & 3 & 1 \end{bmatrix} = -(2)(1)(3)(1) = -6$

Swap rows 1 and 4 to make it lower triangle (need to multiply by -1).

c) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} -3d & -3e & -3f \\ a & b & c \\ g+5a & h+5b & i+5c \end{vmatrix}$

Swap rows 1 and 2 (multiply det by -1). Then multiply new row 1 by -3 (multiply det by -3). Add 5 times row 2 to row 3 (does not change det). Therefore det = 3(-1)(-3) = 9

5. Use Cramer's Rule to solve the system:

$$\begin{cases} 3x + 2y = 7 \\ x - y = 4 \end{cases}$$

$x = \dfrac{\det(A_1)}{\det(A)} = \dfrac{-15}{-5} = 3 \quad y = \dfrac{\det(A_2)}{\det(A)} = \dfrac{5}{-5} = -1$

6. Determine the value(s) of k for which the matrix is invertible:

$\begin{bmatrix} -1 & 0 & 1 \\ 1 & k & -1 \\ -1 & 1 & 1 \end{bmatrix}$. det is always 0 for all k, so no k value will make this invertible.

7. Suppose that A is an invertible nxn matrix. Prove that $\det(adj(A)) = (\det(A))^{n-1}$.

$$A^{-1} = \frac{1}{\det(A)} adj(A) \Rightarrow$$

$$adj(A) = \det(A) A^{-1} \Rightarrow$$

$$\det(adj(A)) = (\det(\det(A)A^{-1}) = (\det A)^n \det(A^{-1}) = \frac{(\det A)^n}{\det(A)} = (\det A)^{n-1}$$